M2201 Sheet 1

Seamus O'Shea

Questions 1, 2, 3, 4 and 5 are assessed. Homework due on Thursday 11th of October in the problem class.

1. Let
$$f(x) = 6x^3 + 16x^2 + 5x - 2, \quad g(x) = 2x^2 + 6x + 3$$
be polynomials in $\mathbb{Q}[x]$. Calculate $\gcd(f, g)$ and find h and k such that
$$\gcd(f, g) = hf + kg$$

2. Let
$$f(x) = x^5 - 4x^3 + x^2 + 4x + 4, \quad g(x) = x^3 + x^2 - x + 2$$
Calculate $\gcd(f, g)$ and find h and k such that
$$\gcd(f, g) = hf + kg$$
in: (i) $\mathbb{Q}[x]$, (ii) $\mathbb{F}_3[x]$.

3. A polynomial $f \in \mathbb{Q}[x]$ has degree 3 and is dividible by $x^2 - 1$. The remainder of the Euclidean division of f by $x + 2$ is 7, and the remainder of the Euclidean division of f by $x - 2$ is 15. Find f.

   (Hint. Notice that the general form of f is $f(x) = (ax+b)(x^2-1)$; use the conditions to find a and b.)

4. Let $f \in \mathbb{R}[x]$ be a polynomial of odd degree. Use the intermediate value theorem of analysis to show that f has at least one root in $\mathbb{R}$.

5. Let $f \in \mathbb{R}[x]$ be a monic irreducible polynomial of degree 2. Show that f may be written
$$f(x) = (x - a)^2 + b^2$$
for some $a, b \in \mathbb{R}$ with $b \neq 0$. Conversley, show that any such polynomial is irreducible over $\mathbb{R}$.

6. This question is NOT assessed.

   Let $f(x) = x^n + \ldots + a_0 \in \mathbb{C}[x]$ be a complex polynomial of degree n. If α is a root of f, show that $|\alpha| \leq n \cdot \max_i(a_i)$.

   (Hint. Write $-\alpha^n = a_{n-1}\alpha^{n-1} + \ldots + a_0$; if $|\alpha| > n \cdot \max_i(a_i)$, divide by α^n and take absolute values.)

M2201 Sheet 2

Seamus O'Shea

ALL questions are assessed. Homework due on Thursday 18th of October.

1. Show that x^2+x+1 is irreducible in $\mathbb{F}_2[x]$, and that x^2+x+1 is the only irreducible of degree 2 in $\mathbb{F}_2[x]$.

   Show that x^4+x+1 is irreducible in $\mathbb{F}_2[x]$.

2. Factorize the polynomial x^3-2x^2+4x-8 into irreducibles in $\mathbb{C}[x]$, $\mathbb{R}[x]$, $\mathbb{F}_2[x]$, $\mathbb{F}_3[x]$ and $\mathbb{F}_5[x]$. Justify why your factors are irreducible!

3. Let V be a vector space over a field k. In each case below, say whether the statement is TRUE or FALSE. Provide a proof or a counter-example accordingly.

   (a) Let v_1, v_2, v_3 be three vectors in V. Suppose that any pair $\{v_i, v_j\}$ is lineraly independent. Then $\{v_1, v_2, v_3\}$ is linearly independent.

   (b) Let $v_1, \ldots, v_n$ be a collection of vectors in V. If no v_i is a linear combination of the others, then the family $\{v_1, \ldots, v_n\}$ is linearly independent.

   (c) If U, $W \subset V$ are two subspaces of V, then $\dim(U+W) = \dim(U) + \dim(W)$, where $U+W$ is the internal sum of U and W.

   (d) Let W be another k-vector space. Then $\dim(V \oplus W) = \dim(V) + \dim(W)$, where $V \oplus W$ denotes the direct sum of V and W.

4. Let $f(x) = x^4 - 1$, $g(x) = x^4 - x \in \mathbb{R}[x]$. Let $V = \mathbb{R}[x]_3$ (i.e. V is the $\mathbb{R}$-vector space consisting of real polynomials of degree less than or equal to 3). Let $T\colon V \longrightarrow V$ be the map such that for $h \in V$, $T(h)$ is the remainder of the Euclidean division of fh by g.

   (a) Show that T is a linear map.

   (b) Factorize f and g in $\mathbb{R}[x]$.

   (c) Let $l \in \mathrm{Ker}(T)$; show that $l \in \mathrm{Span}\{x(x^2+x+1)\}$. Now show that $\mathrm{Ker}(T) = \mathrm{Span}\{x(x^2+x+1)\}$.

   (d) Find $\gcd(f, g)$.

PLEASE TURN OVER

(e) Consider the following subspace of $\mathbb{R}[x]_3$:

$$W = \{h \in \mathbb{R}[x]_3 : h(1) = 0\} = \{h \in \mathbb{R}[x]_3 : (x-1)|h\}$$

Prove that $\{x-1, x(x-1), x^2(x-1)\}$ generates W.

(f) Using Bézout's identity, show that $W \subset \mathrm{Im}(T)$. Determine $\dim(\mathrm{Im}(F))$ (use the Rank-Nullity theorem) and conclude that $W = \mathrm{Im}(T)$.

MATH2201 Sheet 3

Seamus O'Shea

October 17, 2012

Homework due Thursday 25th October. All questions are assesed.

1. Let $V = k[x]_2$ have basis $B = \{1, x, x^2\}$, and let $T : V \to V$ be given by $T(f) = 2f + f'$, where f' is the derivative of f. Find $[T]_B$.

2. Let $M_n(k)$ be the vector space of $n \times n$ matrices with coefficients in k. For $A = (a_{ij})_{1 \leq i,j \leq n} \in M_n(k)$, $Tr(A) = \sum_{i=1}^n a_{ii}$ denotes the trace of M. What is the dimension of the following subspace of $M_n(k)$:
$$\{A \in M_n(k) : Tr(A) = 0\}$$
(*Hint: Use the rank-nullity theorem.*)

3. Let V be a vector space over k of dimension n and let $T \colon V \longrightarrow V$ be a linear map. Show that the following are equivalent:
   (a) $\ker(T) = \operatorname{im}(T)$
   (b) $T^2 = 0$ and $n = 2\dim(\operatorname{im}(T))$

4. Let $\theta \in [0, 2\pi)$, and define the rotation matrix $R_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \in M_2(\mathbb{R})$. Show that R_θ does not have any real eigenvalues unless $R_\theta = \pm I_2$. (*Hint: find the characteristic polynomial of R_θ.*)

5. Let $A = (a_{ij})_{1 \leq i,j \leq n}$ be an $n \times n$ matrix over k such that, for each $1 = 1, \ldots, n$ we have
$$\sum_{j=1}^{n} a_{ij} = 0.$$
Show that 0 is an eigenvalue of A (i.e. 0 is an eigenvalue of the linear map T_A associated to A.)

6. Let V be a vector space over k of dimension n, and let $T : V \to V$ be a linear map. Suppose that V has a basis consisting of eigenvectors $B = \{v_1, \ldots, v_n\}$ with *distinct* eigenvalues $\lambda_1, \ldots, \lambda_n$ (this happens, for example, when $[T]_B = \mathrm{Diag}(\mu_1, \ldots, \mu_n)$ is diagonal with $\mu_i \neq \mu_j$ for $i \neq j$). Show that any eigenvector $v \in V$ is of the form $v = av_i$ for some $1 \leq i \leq n$ and some $a \in k$.

7. Let $A \in M_n(\mathbb{C})$; we say that A is *nilpotent* if $A^r = 0$ for some r.
   (a) Show that if A is nilpotent, then $I_n - A$ is invertible.
   (b) Show that if A is nilpotent, then every eigenvalue of A is equal to 0. Hence show that A is nilpotent if and only if $\mathrm{ch}_A(x) = x^n$.

MATH2201 Sheet 4

Seamus O'Shea

October 23, 2012

Questions 1, 2, 3, 4a and 4b are assessed. Homework due Thursday 1st November.

1. Let $T : k[x]_n \to k[x]_n$ be the linear map given by $T(f) = f + f'$, where f' denotes the derivative of f. Show that T is not diagonalizable for $n \geq 1$.

2. Let $A = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ -3 & -1 & 4 \end{pmatrix} \in M_2(\mathbb{R})$. Find all eigenvalues of A, and find bases for $V_1(\lambda), V_2(\lambda), \ldots$ for each eigenvalue λ of A.

3. Let V be the $\mathbb{C}$-vector space given by $V = \{a\sin(x) + b\cos(x) + cx\sin(x) + dx\cos(x) \mid a, b, c, d \in \mathbb{C}\}$. Let $D : V \to V$ be the linear map given by $D(f) = f'$, where f' denotes the derivative of f. Calculate the $\mathrm{ch}_D(x)$, and hence find all complex eigenvalues of D. Find $m_D(x)$ and find a basis for $V_1(\lambda)$ for each eigenvector λ of D.

4. (a) For any $a, b, c \in \mathbb{C}$ let
$$A = \begin{pmatrix} a & c & b \\ b & a & c \\ c & b & a \end{pmatrix}.$$
(Such a matrix is known as a circulant matrix.) If $\omega = e^{2\pi i/3}$ (so that $1 + \omega + \omega^2 = 0$), show that $v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ \omega \\ \omega^2 \end{pmatrix}$ and $v_3 = \begin{pmatrix} 1 \\ \omega^2 \\ \omega \end{pmatrix}$ are eigenvectors of A, and find λ_j such that $Av_j = \lambda_j v_j$ for $j = 1, 2, 3$.

   (b) Is A always diagonalizable? Either give a proof that A is diagonalizable, or provide a counter-example.

   (c) (NOT ASSESSED) Let $P = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}$. Find P^{-1} (use the formula $P^{-1} = (1/\det(P)) \cdot \mathrm{adj}(P)$, where $\mathrm{adj}(P)$ is the adjoint matrix of P) and verify that $P^{-1}AP = \mathrm{Diag}(\lambda_1, \lambda_2, \lambda_3)$ (where A is the matrix from part (a)).

   (d) (NOT ASSESSED) Generalize the arguments of part (a) to show that if
$$A = \begin{pmatrix} a_1 & a_n & \ldots & a_2 \\ a_2 & a_1 & \ldots & a_3 \\ \vdots & & \ddots & \vdots \\ a_{n-1} & a_{n-2} & \ldots & a_n \\ a_n & a_{n-1} & \ldots & a_1 \end{pmatrix} \in M_n(\mathbb{C}),$$

then $\{v_i = \begin{pmatrix} 1 \\ \zeta^k \\ \zeta^{2k} \\ \vdots \\ \zeta^{nk} \end{pmatrix} \mid k = 0, 1, \ldots, n-1\}$ is a set of eigenvectors for A, where $\zeta = e^{2\pi i/n}$. By considering the eigenvalues λ_k associated to v_k, show that

$$\det(A) = \prod_{k=0}^{n-1} \sum_{j=1}^{n} a_{n-j+1} \zeta^{jk}.$$

MATH2201 Sheet 5

Seamus O'Shea

November 7, 2012

Questions 1, 2, and 3 are assessed. Homework due Thursday 15th November (after reading week).

1. For each of the following cases, either find the Jordan normal form of $T : V \to V$, or state that not enough information has been given to determine the JNF of T:

   (a) $\mathrm{ch}_T(x) = (x+4)^6$, $m_T(x) = (x+4)^5$;
   (b) $\mathrm{ch}_T(x) = (x-2)^4$, $m_T(x) = (x-2)^2$;
   (c) $\mathrm{ch}_T(x) = (x+1)^4$, $m_T(x) = (x+1)$;
   (d) $\mathrm{ch}_T(x) = (x+1)^4$, $\dim(V_1(-1)) = 3$;
   (e) $\mathrm{ch}_T(x) = (x+1)^5$, $\dim(V_1(-1)) = 3$;
   (f) $\mathrm{ch}_T(x) = (x-3)^7$, $m_T(x) = (x-3)^4$, $\dim(V_1(3)) = 3$;
   (g) $\mathrm{ch}_T(x) = (x-3)^9$, $m_T(x) = (x-3)^4$, $\dim(V_1(3)) = 4$;
   (h) $\mathrm{ch}_T(x) = (x-6)^8$, $\dim(V_1(6)) = 4$, $\dim(\mathrm{im}((T - 6\cdot\mathrm{id})^2)) = 1$.

2. Let $V = \mathbb{C}[x]_{90}$ be the $\mathbb{C}$-vector space consisting of all polynomials of degree ≤ 90. Let $D : V \to V$ be the linear map given by
$$D(f) = \frac{d^3 f}{dx^3}.$$
Show that $\mathrm{ch}_D(x) = x^{91}$, and find $m_D(x)$. Clearly
$$B = \{\frac{x^n}{n!} \mid 0 \leq n \leq 90\}$$
is a basis for V; divide B into a series of sets B_i such that $B_1 \cup \ldots \cup B_i$ is a basis for the generalized eigenspaces $V_i(0)$, and show that $D(B_i) \subset B_{i-1}$, so that B is a Jordan basis for V. Hence find the Jordan normal form of D (written in terms of Jordan blocks).

3. For each of the following matrices A, determine whether A is diagonalizable over (a) $\mathbb{Q}$, (b) $\mathbb{R}$, (c) $\mathbb{C}$.
$$\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 5 & 9 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} 2 & -3 \\ 1 & -1 \end{pmatrix}$$

4. (NOT ASSESSED) For each of the following matrices over $\mathbb{C}$, find (a) the characteristic and minimal polynomials, (b) a Jordan basis, (c) the Jordan normal form.
$$\begin{pmatrix} 4 & 3 & 2 \\ -4 & -3 & -3 \\ 1 & 1 & 2 \end{pmatrix}, \begin{pmatrix} -1 & -2 & -1 \\ 3 & 4 & 1 \\ 3 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 \\ 1 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

MATH2201 Sheet 6

Seamus O'Shea

November 14, 2012

All questions are assessed. Homework due on Thursday 22nd of November.

1. Let V be a vector space of dimension $n < \infty$ over k, and let $f \in V^*$ be a non-zero linear form. Show that $\ker(f)$ is a vector subspace of V, and $\dim(\ker(f)) = n - 1$. Hence show that, if g is another linear form on V, with $\ker(g) = \ker(f)$, then $g = \lambda f$ for some $\lambda \in k$.

2. Let V be a vector space over k, with subspace W. The **annihilator** of W is the set

$$W^\circ = \{f \in V^* \mid f(w) = 0 \text{ for all } w \in W\}.$$

   (a) Show that W° is a subspace of V^*. Show that, if $\dim(V) = n < \infty$, then

$$\dim(W) + \dim(W^\circ) = n.$$

   Hint: first choose a basis $\{w_1, \dots, w_r, v_1, \dots, v_{n-r}\}$ for V such that $\{w_1, \dots, w_r\}$ is a basis for W. Then construct a basis for W° using this.

   (b) Show that, if U is another subspace of V, then

$$(U + W)^\circ = U^\circ \cap W^\circ \text{ and } (U \cap W)^\circ = U^\circ + W^\circ.$$

3. Let $V = \mathbb{R}[x]_2$ be the vector space of polynomials of degree less than or equal to 2. Define three linear forms $\alpha_i : V \to \mathbb{R}$ for $i = 1, 2, 3$ by

$$\alpha_1(f) = \int_0^1 f(x)dx, \;\; \alpha_2(f) = \int_0^2 f(x)dx, \;\; \alpha_3 = \int_{-1}^0 f(x)dx.$$

   Find a basis $\mathcal{B} = \{f_1, f_2, f_3\}$ for V such that $\mathcal{B}^* = \{\alpha_1, \alpha_2, \alpha_3\}$, and conclude that $\{\alpha_1, \alpha_2, \alpha_3\}$ is a basis for V^*.

   Hint: first find an anti-derivative for f_1, f_2 and f_3, using the equations

$$\alpha_i(f_j) = \begin{cases} 1 \; \textit{if } i = j, \\ 0 \; \textit{if } i \neq j. \end{cases}$$

   Remember that antiderivatives are only determined up to addition of a constant!

M2201 Sheet 7

Seamus O'Shea

All questions are assessed. Homework due on Thursday 29th of November.

1. Find the real and complex canonical forms of the following quadratic forms:

$$q_1\begin{pmatrix}x\\y\\z\end{pmatrix} = x^2+3y^2+z^2+2xy-2xz-2yz,$$

$$q_2\begin{pmatrix}x\\y\\z\end{pmatrix} = 4x^2+4xz+2yz,$$

$$q_3\begin{pmatrix}x\\y\\z\end{pmatrix} = -8x^2+6xy-y^2+12xz-4yz-5z^2.$$

   Which of q_1, q_2 and q_3 are equivalent over (i) $\mathbb{R}$, (ii) $\mathbb{C}$?

2. Let $V=\mathbb{R}[x]_n$ be the real vector space of all real polynomials of degree $\leq n$. If

$$\alpha(f,g)=\int_0^1 xf(x)g(x)dx,$$

   show that α is a symmetric bilinear form on V. By considering $\alpha(f,f)$, where f is any element of V, show that the (real) canonical form of α is I_{n+1}.

3. Let $\mathbb{F}_3=\{0,1,2\}$ be the field of integers modulo 3. Show by double operations that the following matrices over $\mathbb{F}_3$ are congruent:

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\quad \begin{pmatrix}1&0&0\\0&2&0\\0&0&2\end{pmatrix}$$

   Show that $\begin{pmatrix}1&0&0\\0&1&0\\0&0&2\end{pmatrix}$ is not congruent to $\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$ over $\mathbb{F}_3$. Hence show that $\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}$ is not congruent to $\begin{pmatrix}1&0&0\\0&2&0\\0&0&0\end{pmatrix}$ over $\mathbb{F}_3$. How many distinct canonical forms are there for quadratic forms $q:\mathbb{F}_3^3\to\mathbb{F}_3$?

M2201 Sheet 8

Seamus O'Shea

ALL questions are assessed. Homework due on Thursday 6th of December.

1. Let $V = M_n(\mathbb{C})$ and define $\langle A, B\rangle = \mathrm{tr}(A\bar{B}^T)$. Show that $(V, \langle -, -\rangle)$ is a complex inner product space, and find an orthonormal basis for V. Show that for any matrix $A = (a_{i,j})_{1\leq i,j\leq n} \in M_n(\mathbb{C})$, we have

$$|\mathrm{tr}(A)|^2 \leq n \cdot \sum_{1\leq i,j\leq n} |a_{i,j}|^2$$

   Hint: apply the Cauchy-Schwartz inequality to I_n and A.

2. Find an orthonormal basis for the following inner product spaces:
   (a) $V = \mathrm{Span}\{\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\2\\3\end{pmatrix}\} \subset \mathbb{C}^3$, with $\langle v, w\rangle = v^T\bar{w}$.
   (b) $V = \mathbb{R}[x]_2$, with $\langle f, g\rangle = \int_0^1 f(x)g(x)dx$.

3. (a) Let V be a inner product space. Show that, if $x, y \in V$, then

$$||x+y||^2 + ||x-y||^2 = 2||x||^2 + 2||y||^2.$$

   This identity is known as the parallelogram law.

   (b) We say that a sequence of vectors $x_n \in V$ is a Cauchy sequence if, for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that, if $n, m \geq N$, then $||x_n - x_m|| < \varepsilon$.
   Suppose that $||x_n|| \to a$ as $n \to \infty$ for some $a \in \mathbb{R}$. Using the parallelogram law, show that x_n is a Cauchy sequence.
   Hint: Write down the limit of $||x_n||^2$ as $n \to \infty$. Then use the triangle inequality to find the limit of $||x_n + x_m||$ as $\min(n, m) \to \infty$ ...